Indian Statistical Institute, Bangalore Centre.
End-Semester Exam : Topics in Gaussian Processes

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Max. points : 100. **Time Limit : 4 hours.**

Answer as many questions as you can.
Give necessary justifications and explanations for all your arguments. If you are citing results from the notes or class, mention it clearly.

1. For smooth functions f, define the generator $\mathcal{L}$ of a Markov semi-group as
$$\mathcal{L}f := \sum_{i,j=1}^{n} g^{i,j}\partial^2_{i,j}f + \sum_{i=1}^{n} b^i\partial_i f,$$
where $g(x) = (g^{i,j}(x))_{1\leq i,j\leq n}$ and $b(x) = (b^i(x))_{1\leq i\leq n}$ are smooth, respectively $n\times n$ symmetric matrix-valued and $\mathbb{R}^n$-valued functions ox x. Let μ be the stationary measure and P_t be the corresponding Markov semi-group. (**20**)
   (a) Compute the *carré du champ* operator $\Gamma(f,g)$ and show that it satisfies the chain rule $\Gamma(f,\phi\circ g) = \Gamma(f,g)\phi'\circ g$.
   (b) Show that
   $$\mathcal{E}(\log P_t f, P_t f) \leq \sqrt{\mu(\Gamma(f,f)/f)\mu(f\Gamma(P_t\log P_t f, P_t\log P_t f))}.$$
   (c) Show that if the Bakry-Émery criterion with constant c holds for all f, then
   $$\mathcal{E}(\log P_t f, P_t f) \leq e^{-t/c}\sqrt{\mathcal{E}(\log f, f)\mu(f P_t\Gamma(\log P_t f, \log P_t f))}.$$
   (d) Show that if the Bakry-Émery criterion with constant c holds for all f, then the following *log-Sobolev inequality* holds
   $$Ent[f^2] \leq 2c\mathcal{E}(f,f).$$

2. Let $(\mathbb{S}, d)$ be a metric space with a measure μ. Prove the following. (**30**)
   (a) Show that the isoperimetric inequality for $(\mathbb{S},d,\mu)$ with constants C,σ^2 is equivalent to the concentration inequality for Lipschitz functions w.r.t. the median and with constants C,σ^2.
   (b) Show that if isoperimetric inequality for $(\mathbb{S},d,\mu)$ with constants C,σ^2 holds, then the concentration inequality for Lipschitz functions w.r.t. the mean holds with constants $e^{C^2\pi/4}, 4\sigma^2$.

(c) Suppose concentration inequality for Lipschitz functions w.r.t. the mean holds with constants C, σ^2. Then show that concentration inequality for Lipschitz functions w.r.t. the median holds with constants $2C, 4\sigma^2$

3. State which implications among the below items hold. If specific additional conditions are necessary for the implication, please mention the same. (**10**).

   (a) Bakry-Émery criterion.

   (b) Local Poincaré inequality.

   (c) Poincaré inequality.

   (d) Local modified Log-Sobolev inequality.

   (e) Modified Log-Sobolev inequality.

   (f) Log-Sobolev inequality.

   (g) Exponential L_2 ergodicity.

   (h) Exponential entropic ergodicity.

   (i) Sub-Gaussianity for Lipschitz functions with suitable variance proxy.

   (j) Hypercontractivity.

   (k) T_1 inequality.

   (l) T_2 inequality.

   (m) $L^1 - L^2$ inequality.

4. *Gaussian polymer on a complete graph :* Let $X = (X_v)_{v \in \{1,\ldots,n\}^2}$ be i.i.d. $N(0,1)$. Let γ be a path of length n of the form $(1, a_1), (2, a_2), \ldots, (n, a_n)$ where $a_i \in \{1, \ldots, n\}$. Define $H_n(\gamma) := -\sum_{v \in \gamma} g_v$ and $F(X) := \min_\gamma H_n(\gamma)$. Compute $\mathsf{VAR}(F(X))$ and compare it with the variance bound from the Poincare' inequality. (**10**)

5. *Optimal Poincaré inequality :* Let $\mathcal{L}$ be a a generator of a reversible Markov semigroup with stationary distribution μ. (**20**)

   (a) Show that $-\mathcal{L}$ has eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq$ ..... Here eigenvalue is referred to as eigenvalues of $-\mathcal{L}$ as an operator on $L^2(\mu)$.

   (b) Denoting the corresponding orthonormal basis of eigenfunctions as $u_0, \ldots, u_n, \ldots$ show that $u_0 \equiv 1$ i.e., the constant function.

   (c) Using the orthonormal series expansion of f, compute $\mathbb{E}[f], \mathcal{E}(f, f)$ in terms of $\lambda_k, a_k := \mu(u_k f), k = 0, \ldots$.

   (d) Show that if $\lambda_1 > 0$ then $\mathsf{VAR}(f) \leq \lambda_1^{-1} \mathcal{E}(f, f)$ for all f and also that there exists a non-constant f such that the equality holds.

(e) If $\lambda_1 = 0$ show that there exists f such that $0 = \mathcal{E}(f,f) < \mathsf{VAR}(f)$.

(f) Conclude that the Poincaré inequality holds iff $\lambda_1 > 0$ and in that case the optimal constant is λ_1^{-1}.

6. *Birth-death chain :* Let N, N' denote independent $Poi(\lambda)$ random variables. Given any $n \in \mathbb{N}, t > 0$, define $e^{-t} \circ n = \sum_{i=1}^{n} X_{i,n}(t)$ where $X_{i,n}(t)$ are i.i.d. $Ber(e^{-t})$. In other words, $e^{-t} \circ n \stackrel{d}{=} Bin(n, e^{-t})$. Define a Markov process on $\mathbb{N}$ as follows : $G_t := e^{-t} \circ G_0 + (1 - e^{-t}) \circ N'$ where all the involved Bernoulli random variables are independent. (**20**)

   (a) Show that G_t is a reversible with stationary distribution N.

   (b) Show that $\mathcal{L}f(k) := \lambda \Delta f(k+1) - k\Delta f(k)$ where $\Delta f(k) = f(k) - f(k-1)$.

   (c) Show that $\Delta P_t f = e^{-t} P_t \Delta f$

   (d) Show that $\mathbb{E}(F(N)) - F(k) = \int_0^\infty \mathcal{L} P_s F(k) ds$.

   (e) Show that $\mathcal{E}(f,f) = \mathbb{E}(\Delta f(N)^2)$.

7. *Glauber dynamics for Markov chains on hypercube :* Let $\Omega = \{0,1\}^n$ and π be a distribution on Ω. For a configuration $\xi \in \Omega$, let ξ^i denote the configuration with the ith co-ordinate flipped. Recall that the generator $\mathcal{L}$ is defined via the matrix Λ' as $\mathcal{L}f(\xi) = \sum_\eta \Lambda(\xi,\eta) f(\eta)$. (**10**)

   (a) Assume that $\mathcal{L}$ is the generator of a reversible Markov chain on Ω. Show that $\mathcal{E}(f,f) = \frac{1}{2} \sum_{\xi,\eta} \pi(\xi)\pi(\eta)\Lambda(\xi,\eta)(f(\xi) - f(\eta))^2$.

   (b) Define Λ as follows : For $\eta \neq \xi$, $\Lambda(\xi,\eta) = \frac{\pi(\xi)+\pi(\eta)}{\pi(\xi)}$ if $\eta = \xi^i$ for some $1 \leq i \leq n$ and else $\Lambda(\xi,\eta) = 0$. Show that

$$\mathcal{E}(f,f) = \mathbb{E}_\pi[\sum_{i=1}^{n} [f(\xi^i) - f(\xi)]^2].$$

8. *Discrete Second-order Poincaré inequality :* Let $X = (X_1, \ldots, X_n)$ be a random vector of independent random variables. Let $X' = (X_1', \ldots, X_n')$ be an independent copy of X. We define X^A as follows : $X_i^A := X_i' 1[i \in A] + X_i 1[i \notin A]$. Set $X^j := X^{\{j\}}$ and $\Delta_j f(X) = f(X) - f(X^j)$. Assume ϕ is a smooth function with bounded derivatives below. (**30**)

   (a) Show that $f(X) - f(X') = \sum_{A \subsetneq [n]} \frac{1}{\binom{n}{|A|}(n-|A|)} \sum_{j \notin A} \Delta_j f(X^A)$.

(b) For $A \subsetneq [n], j \notin A$, set $U_{A,j} := U(X_1, \ldots, X_n, X'_1, \ldots, X'_n) := g(X)\Delta_j f(X^A)$. Define $U'_{A,j} := U(X_1, \ldots, X'_j, \ldots, X_n, X'_1, \ldots, X_j, \ldots, X'_n)$. Show

$$\mathbb{E}[U_{A,j}] = \frac{1}{2}\mathbb{E}[U_{A,j} + U'_{A,j}] = \frac{1}{2}\mathbb{E}[\Delta_j g(X)\Delta_j f(X^A)].$$

(c) Show that $\mathsf{Cov}\,(f(X), g(X)) = \sum_{A \subsetneq [n]} \frac{1}{\binom{n}{|A|}(n-|A|)} \sum_{j \notin A} \mathbb{E}[U_{A,j}]$.

(d) For $A \subsetneq [n], j \notin A$, set $R_{A,j} := \Delta_j(\phi \circ f)(X)\Delta_j f(X^A)$ and $\tilde{R}_{A,j} := \phi'(f(X))\Delta_j f(X)\Delta_j f(X^A)$. Show that

$$|R_{A,j} - \tilde{R}_{A,j}| \leq \frac{\|\phi'\|_\infty}{2}(\Delta_j f(X))^2 \Delta_j f(X^A).$$

(e) Define $T := \frac{1}{2}\sum_{A \subsetneq [n]} \frac{1}{\binom{n}{|A|}(n-|A|)} \sum_{j \notin A} \Delta_j f(X)\Delta_j f(X^A)$. Set $W = f(X)$ and assume that $\mathbb{E}[W] = 0, \mathbb{E}[W^2] = 1$. Show that

$$|\mathbb{E}[\phi(W)W] - \mathbb{E}[\phi'(W)T]| \leq \frac{\|\phi'\|_\infty}{4} \sum_{j=1}^{n} \mathbb{E}[|\Delta_j f(X)|^3].$$

(f) Assume that $\|\phi'\|_\infty \leq 1$ and $\|\phi''\|_\infty \leq 2$. Show that

$$|\mathbb{E}[\phi'(W)] - \mathbb{E}[\phi(W)W]| \leq \sqrt{\mathsf{VAR}(\mathbb{E}[T|W])} + \frac{1}{2}\sum_{j=1}^{n} \mathbb{E}[|\Delta_j W|^3].$$